I. Matrix transformations

-A transformation $T : R^n \rightarrow R^m$ is called a matrix transformation and can be written as $T(x) = Ax$ with A as an mxn matrix
-A vector $\vec{v}$ is in the range of T if there exists a vector $\vec{w}$, when multiplied with A, that will result in $\vec{w}A = \vec{v}$
-To find a vector that is not in range of T, augment it with a column of variables (ws) then solve for an instance that leads it to be inconsistent.

II. One-to-one and onto transformations

-For $T : R^n \rightarrow R^m$ We Say T is onto if $T(\vec{x}) = \vec{b}$ has a solution for every $\vec{b} \in R^m$
-For $T : R^n \rightarrow R^m$ We Say T is one-to-one if $T(\vec{x}) = \vec{b}$ has at most one solution for every $\vec{b} \in R^m$
- T maps onto R^m if every row has a pivot
- T is one-to-one if every column has a pivot

III. Matrix Multiplication

given A = $\begin{bmatrix} 3 & -5 \\ 2 & 1 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 3 & 0 \\ 4 & 2 & -3 \end{bmatrix}$

A*B = $\begin{bmatrix} 3 & -5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} ... \begin{bmatrix} 3 & -5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} ... \begin{bmatrix} 3 & -5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

A*B = $\begin{bmatrix} -17 \\ 6 \end{bmatrix} ... \begin{bmatrix} -1 \\ 8 \end{bmatrix} ... \begin{bmatrix} 15 \\ -3 \end{bmatrix}$

A*B = $\begin{bmatrix} -17 & -1 & 15 \\ 6 & 8 & -3 \end{bmatrix}$

A. Rules

AB != BA
if AB = AC, B != C (unless they are equal)
AI = A = IA
even if A and B are non-zero matrices AB = 0 can occur
Transpose rules
$(A^T)^T = A$
$(A + B)^T = A^T + B^T$
$(AB)^T = B^T * A^T$

IV. Inverses

- for a 2x2 matrix: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- for an nxn matrix: Augment the original matrix with the identity matrix, row reduce until you get the identity matrix on the left side, the right side will be the inverse of the original matrix

A. Invertible matrix theorem

- A is an nxn matrix
- A is invertible
- A is row-equivalent to the identity matrix
- Ax = 0 has only the trivial solution
- the columns of A are linearly independent
- The transformation $T(\vec{x}) = A\vec{x}$ is onto
- There is an nxn matrix C so that CA = I
- There is an nxn matrix D so that AD = I
- A^T is invertible

$(A^-1)^{-1}) = A$
$(AB)^{-1} = B^{-1} * A^{-1}$
$(A^T)^{-1} = (A^{-1})^T$

V. Subspaces

A subset H of R^n is called a subspace if:
- $\vec{0} \in H$
- if u_1 and u_2 are in H then $u_1 + u_2$ are also in H
- if u is in H and c is a scalar then cu is also in H

VI. Bases

H is a subspace of R^n. Vectors $v_1, ...v_n$ form a basis for H if:
- Span$\{v_1, ...v_n\}$ = H, ie A pivot in each column
- $v_1, ...v_n$ are linearly independent, ie A pivot in each row

VII. Null Space

The Null Space of a matrix A is the set of all solutions to Ax = 0, dentoted Nul A.
Augment the matrix A with zeros, solve for the parametric form, the resulting vectors of coefficients is the basis for the null space.

VIII. Column Space

The column space of a matrix A is the set of all linear combinations of the columns of A.
Row reduce, find every column with a pivot, the corresponding columns in the original matrix form the basis for the column space.

IX. Dimension, Rank and Nullity

If H is a nonzero subspace of R^n then eery basis for H consists of the same number of vectors. The number of vectors that form the basis is called the dimension, denoted dim
The **rank** of A is the dimension of the column space
The **nullity** of A is the dimension of the null space
Rank Nullity theorem states:
let A be an mxn matrix. Let p denote the number of pivots.
Rank of A = p; Nullity of A = n-p; Rank A + Nullity A = n

X. Determinants

A_{ij} is the (n-1)x(n-1) matrix obtained by deleting the ith row and jth column of A. The determinant of A_{ij} is called the minor denoted M_{ij}
The (i,j)-cofactor is $c_{ij} = (-1)^{i+j} * M_{ij}$
The determinant for a 2x2 matrix:
$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, det(A) = ad - bc$
For an nxn matrix, det(A) = $a_{11} * c_{11} + a_{12} * c_{12} + ... + a_{nn} + c_{nn}$
For triangular matrices, the det(A) is equal to the product of the entries along the main diagonal
Properties of determinants:
- Swapping rows results in swapping positivity/negativity
- if B is obtained by adding one row to another of A (even

when multiplying by a scalar), det(A) = det(B)
- if B is the result of scaling a row of A by the scalar k, k*det(A) = det(B)
- The matrix A is invertible if det(A) != 0
- det (A^-1) = $\frac{1}{det(A)}$
- if AB is invertible then A and B are invertible

XI. Determinants and transformations

Suppose $T : R^2 \rightarrow R^2$ is a linear transformation. let S be a region in R^2 with finite area. If A is the standard matrix for T, then (Area of T(S)) = $|det(A)|$ * (Area of S)
Example: Area of the elipse
The ellipse is the result of stretching the unit circle (circle with radius=1) by a factor ofa in the x-direction and b in the y-direction, which can be defined by $T([x_1, x_2]) = [ax_1, bx_2]$
A = $[T([1, 0])T([0, 1])] = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$
by the Theorem, (Area of the ellipse) = (area of s1) * $|det(A)|$
(Area of the ellipse) = $\Pi(r = 1)^2 * |det(A)| = \Pi * 1 * |ab|$
(Area of the ellipse) = $\Pi * ab$

XII. Eigenvectors and eigenvalues

- A scalar λ is called an eigenvalue of T if T(x) = λx has a nontrivial solution
-Each nontrivial solution to T(x) = λx is called an eigenvector associated to λ
- The set of all solutions to T(x) = λx, is the eigenspace of T associated to λ

Let A be an nxn matrix and λ be any scalar
- λ is and eigenvalue of A if $(A - \lambda I_n)x = 0$ has a nontrivial solution
- v is an eigenvector associated to λ if v != 0 and $(A - \lambda I_n)v = 0$
-$E_\lambda(A) = Nul(A - \lambda I_n)$

XIII. Characteristic Polynomial

Let A be an nxn matrix
-The polynomial $p(\lambda) = det(A - \lambda I)$ is called the characteristic polynomial
-The equation $det(A - \lambda I) = 0$ is call the characteristic equation of A
-The eigenvalues of A are the roots of the characteristic polynomial
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
-A and B are said to be similar when they have the same characteristic polynomial
-if they are similar they have the same determinant, same ability to be inverted, same characteristic polynomial, and same eigenvalues

XIV. Diagonalization

- A Matrix A is Diagonalizable if $A = PDP^{-1}$ for some diagonal matrix D and invertible matrix P
- A is Diagonalizable if all columns are linearly independent eigenvectors: $P = [\vec{v_1} ... \vec{v_n}]$ D (for 2x2) = $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ - If A is nxn and has n eigenvectors, A is diagonalizable

A. *For Symmetric Matrices*

- Symmetric $A = A^T$
- for matrix U, it is orthogonal if $U^T U = I_n$ or $U^T = U^{-1}$
- matrix U has orthonormal columns if and only if $U^T U = I_n$
- An nxn matrix A is said to be orthogonally diagonalizable if there are an orthogonal matrix P and a diagonal matrix D such that:
$A = PDP^{-1} = PDP^T$ - For finding $A^n, A^n = PD^n P^{-1}$

XV. Markov Chains

-A vector v is called a **probability vector** if the entries of v are non-negative and add to 1
- A square matrix T is called a stochastic matrix if every column of A is a probability vector

XVI. Length, distance, and orthagonality

-The length or norm of a vector is $||v|| = \sqrt{v * v}$
-The distance between two vectors is:
$||u - v|| = \sqrt{(u_1 - v_1)^2 + ... + (u_n - v_n)^2}$
- if θ is the angle between u and v, u $u \cdot v = ||u|| \cdot ||v|| \cos\theta$
- u and v are orthogonal if $u \cdot v = 0$

XVII. Orthogonal Projections

-Let L = Span{u} for some nonzero u in R^n. for any y in R^n, define the orthogonal projection of y onto L be : $proj_L(y) = (\frac{y \cdot u}{u \cdot u})u$
- A set of vectors $v_1, ..., v_k$ is is said to be to be orthogonal if each pair of vectors in the set is orthogonal. If the set is orthogonal and every vector is a unit vector, the set is said to be orthonormal.

-Let W be a subspace of R^n and let $u_1, ..., u_k$ be any orthogonal basis for W. For any y in R^n, the orthogonal projection of y onto W is: $proj_W(y) = (\frac{y \cdot u_1}{u_1 \cdot u_1})u_1 + ... + (\frac{y \cdot u_k}{u_k \cdot u_k})u_k$

A. *Gram-Schmidt*

given a basis $x_1, x_2, ...x_p$ for nonzero subspace W, define: $v_1 = x_1; v_2 = x_2 - (\frac{x_2 \cdot v_1}{v_1 \cdot v_1})v_1; ...v_p = x_p - (\frac{x_p \cdot v_1}{v_1 \cdot v_1})v_1 - ... - (\frac{x_p \cdot v_{p-1}}{v_{p-1} \cdot v_{p-1}})v_{p-1}$

B. *Least Squares*

If Ax = b has no solutions, $\hat{x}$ is the closest solution to it, using: $A^T Ax = A^T b$